

A Discussion on a Limit Theorem and Its Application

YUAN Nan-Qiao

(Dept. of Math, Daxian Teachers College, Dazhou 635000, China)

Abstract: This paper proposes that a limit theorem can help to solve a specific limit problem of sum formula and that some limit of product formula can also be solved by exploiting the feature of logarithm function.

Key words: limit theorem; sum formula; product formula

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In calculus, we will usually solve a specific limit problem of sum formula

$$X_n = \frac{1}{n^3 + 1} + \frac{2^2}{n^3 + 2^2} + \cdots + \frac{n^2}{n^3 + n^2}.$$

But this sum formula can't sum directly, and it can't change into some kinds of function's integral sum. So it is hard to work out its limit, for solve this problem.

This paper's proposes is that a limit theorem can help to solve this limit problem of sum formula and that some limit of product formula can also be solved by logarithm function.

Theorem 1 Let (a) f be differentiable at $x=0$ and $f(0)=0$, (b) g be integrable for $x \in [a, b]$. We have

$$\lim_{\|T\| \rightarrow 0} \sum_{k=1}^n f[g(\xi_k) \Delta x_k] = f(0) \int_a^b g(x) dx,$$

where $T: a = x_1 < x_2 < \cdots < x_n = b$, $\xi_k \in [x_{k-1}, x_k]$, $\Delta x_k = x_k - x_{k-1}$, $\|T\| = \max_{1 \leq k \leq n} \{\Delta x_k\}$.

Proof By the (a), for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|x| < \delta$ implies $|f(x) - f(0)x| < \varepsilon|x|$.

Then by the (b), there exists a real number $M > 0$ such that $|g(x)| \leq M$ for $x \in [a, b]$ and there is a $\delta_2 > 0$ such that $\|T\| < \delta_2$ implies

$$\left| \sum_{i=1}^n g(\xi_i) \Delta x_i - \int_a^b g(x) dx \right| < \varepsilon.$$

Let $\delta = \min \left\{ \delta_1, \frac{\delta_2}{M} \right\}$, so when $\|T\| < \delta$, we get

$$\begin{aligned} & \left| \sum_{i=1}^n f[g(\xi_i) \Delta x_i] - f(0) \int_a^b g(x) dx \right| \\ & \leq \left| \sum_{i=1}^n f[g(\xi_i) \Delta x_i] - f(0) \sum_{i=1}^n g(\xi_i) \Delta x_i \right| + |f(0)| \left| \sum_{i=1}^n g(\xi_i) \Delta x_i - \int_a^b g(x) dx \right| \\ & \leq \varepsilon |f(0)| + \sum_{i=1}^n |f[g(\xi_i) \Delta x_i] - f(0) g(\xi_i) \Delta x_i| \\ & \leq \varepsilon |f(0)| + \sum_{i=1}^n \varepsilon g(\xi_i) \Delta x_i \quad \cdots (\text{因为 } |g(\xi_i) \Delta x_i| < \delta) \\ & \leq \varepsilon |f(0)| + \varepsilon M \sum_{i=1}^n \Delta x_i = \varepsilon [|f(0)| + M(b-a)] \end{aligned}$$

and therefore

$$\lim_{\|T\| \rightarrow 0} \sum_{i=1}^n f[g(\xi_i) \Delta x_i] = f(0) \int_a^b g(x) dx.$$

We note the preceding argument was based on the assumption that $f(0) = 0$. For the case that $f(0) \neq 0$. We can show that $\lim_{\|T\| \rightarrow 0} \sum_{i=1}^n f[g(\xi_i) \Delta x_i] = +\infty$ for $f(0) > 0$ and $\lim_{\|T\| \rightarrow 0} \sum_{i=1}^n f[g(\xi_i) \Delta x_i] = -\infty$ for $f(0) < 0$.

Let $f(x) = x$ then theorem 1 has become

$$\lim_{\|T\| \rightarrow 0} \sum_{i=1}^n g(\xi_i) \Delta x_i = \int_a^b g(x) dx.$$

This is definition of definite integral, and by logarithm function we get

Corollary 2 If f be differentiable at $x=0$ and $f(0) = 1$ and g be integrable for x into $[a, b]$ then we have

$$\lim_{\|T\| \rightarrow 0} \prod_{k=1}^n f[g(\xi_k) \Delta x_k] = \exp \left\{ f'(0) \int_a^b g(x) dx \right\}.$$

In practical is usually divide $[0, 1]$ into n parts, and choose $\xi_k = \frac{k}{n}$ ($k=1, 2, \dots, n$).

Corollary 3 Let f be differentiable at $x=0$ and g be integrable for x into $[0, 1]$, then we have

(a) If $f(0) = 0$, we have

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f \left[\frac{1}{n} g \left(\frac{k}{n} \right) \right] = f'(0) \int_0^1 g(x) dx.$$

(b) If $f(0) = 1$, we have

$$\lim_{n \rightarrow \infty} \prod_{k=1}^n f \left[\frac{1}{n} g \left(\frac{k}{n} \right) \right] = \exp \left\{ f'(0) \int_0^1 g(x) dx \right\}.$$

Proof By that theorem 1 and logarithm function, we get

$$\begin{aligned} \lim_{\|T\| \rightarrow 0} \prod_{k=1}^n f[g(\xi_k) \Delta x_k] &= \lim_{\|T\| \rightarrow 0} \exp \left\{ \sum_{k=1}^n \ln f[g(\xi_k) \Delta x_k] \right\} \\ &= \exp \left\{ \lim_{\|T\| \rightarrow 0} \sum_{k=1}^n \ln f[g(\xi_k) \Delta x_k] \right\} = \exp \left\{ [\ln f(x)]'_{x=0} \int_a^b g(x) dx \right\} \\ &= \exp \left\{ \frac{f'(0)}{f(0)} \int_a^b g(x) dx \right\} = \exp \left\{ f'(0) \int_a^b g(x) dx \right\}. \end{aligned}$$

Example 1 Evaluate each of the following:

$$(a) \lim_{n \rightarrow \infty} \left[\frac{1^2}{n^4 + 1^2} + \frac{2^2}{n^4 + 2^2} + \dots + \frac{n^2}{n^4 + n^2} \right].$$

$$(b) \lim_{n \rightarrow \infty} \left[\frac{1^2}{n^3 + 1^2} + \frac{2^2}{n^3 + 2^2} + \dots + \frac{n^2}{n^3 + n^2} \right].$$

$$(c) \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\sqrt[3]{1 + \frac{k}{n^2}} - 1 \right).$$

$$(d) \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\sin \frac{ka}{n^2} \right).$$

$$(e) \lim_{n \rightarrow \infty} \sum_{k=1}^n (a^{\frac{k}{n^2}} - 1) \quad (a > 0)$$

Solution (a) Rewrite the sum in the equivalent form

$$\sum_{k=1}^n \frac{k^2}{n^4 + k^2} = \sum_{k=1}^n \left(\frac{\frac{k}{n}}{1 + \left(\frac{k}{n} \right)^2} \right)^2 \cdot \left[1 + \left(\frac{k}{n} \right)^2 \right].$$

Let $f(x) = \frac{x^2}{1+x^2}$ and $g(x) = x$. Then

$$f'(0) = 0 \quad \text{and} \quad \int_0^1 g(x) dx = \frac{1}{2}.$$

So that by theorem 1,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left[\frac{1^2}{n^4 + 1^2} + \frac{2^2}{n^4 + 2^2} + \cdots + \frac{n^2}{n^4 + n^2} \right] &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{k}{n} \cdot \frac{1}{n} \right)^2 \left/ \left[1 + \left(\frac{k}{n} \cdot \frac{1}{n} \right)^2 \right] \right. \\ &= f'(0) \int_0^1 g(x) dx = 0. \end{aligned}$$

(b) Rewrite the sum in the equivalent form

$$\sum_{k=1}^n \frac{k^2}{n^3 + k^2} = \sum_{k=1}^n \left[\left(\frac{k}{n} \right)^2 \cdot \frac{1}{n} \right] \left/ \left[1 + \left(\frac{k}{n} \right)^2 \cdot \frac{1}{n} \right] \right.$$

Let $f(x) = \frac{x}{1+x}$ and $g(x) = x^2$. Then

$$f'(0) = 1 \quad \text{and} \quad \int_0^1 g(x) dx = \int_0^1 x^2 dx = \frac{1}{3}.$$

So that by theorem 1,

$$\lim_{n \rightarrow \infty} \left[\frac{1^2}{n^3 + 1^2} + \frac{2^2}{n^3 + 2^2} + \cdots + \frac{n^2}{n^3 + n^2} \right] = \frac{1}{3}.$$

(c) Let $f(x) = \sqrt[3]{1+x} - 1$ and $g(x) = x$. Then

$$f'(0) = \frac{1}{3} \quad \text{and} \quad \int_0^1 g(x) dx = \frac{1}{2}.$$

So that by theorem 1,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\sqrt[3]{1 + \frac{k}{n^2}} - 1 \right) = f'(0) \int_0^1 g(x) dx = \frac{1}{6}.$$

(d) Let $f(x) = \sin ax$ and $g(x) = x$. Then

$$f'(0) = a \quad \text{and} \quad \int_0^1 g(x) dx = \frac{1}{2}.$$

So that by theorem 1,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\sin \frac{ka}{n^2} \right) = f'(0) \int_0^1 g(x) dx = \frac{a}{2}.$$

(e) Let $f(x) = a^x - 1$ and $g(x) = x$. Then

$$f'(0) = \ln a \quad \text{and} \quad \int_0^1 g(x) dx = \frac{1}{2}.$$

So that by theorem 1,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n (a^{\frac{k}{n^2}} - 1) = \frac{1}{2} \ln a.$$

Example 2 Evaluate the following limits:

$$(a) \lim_{n \rightarrow \infty} \prod_{k=1}^n \left(1 + \frac{k}{n^2} \right).$$

$$(b) \lim_{n \rightarrow \infty} \prod_{k=1}^n \frac{n^2 + k}{n^2 - k}.$$

$$(c) \lim_{n \rightarrow \infty} \prod_{k=1}^n \frac{(n+k)a}{(n+k)a-1} \quad (a \neq 0).$$

$$(d) \lim_{n \rightarrow \infty} \prod_{k=1}^n \cos \frac{ka}{\sqrt{n}}.$$

Solution (a) We can change the product into an equivalent form by writing

$$\prod_{k=1}^n \left(1 + \frac{k}{n^2} \right) = \prod_{k=1}^n \left(1 + \frac{k}{n} \cdot \frac{1}{n} \right).$$

Let $f(x) = 1+x$ and $g(x) = x$. Then

$$f'(0) = 1 \quad \text{and} \quad \int_0^1 g(x) dx = \frac{1}{2}.$$

So that by corollary 2,

$$\lim_{n \rightarrow \infty} \prod_{k=1}^n \left(1 + \frac{k}{n^2} \right) = e^{\frac{1}{2}}.$$

(b) Rewrite the product in the equivalent form

$$\prod_{k=1}^n \frac{n^2 + k}{n^2 - k} = \prod_{k=1}^n \frac{1 + \frac{k}{n} \cdot \frac{1}{n}}{1 - \frac{k}{n} \cdot \frac{1}{n}}.$$

Let $f(x) = \frac{1+x}{1-x}$ and $g(x) = x$. Then

$$f'(0) = 2 \quad \text{and} \quad \int_0^1 g(x) dx = \frac{1}{2}.$$

So that by corollary 2,

$$\lim_{n \rightarrow \infty} \prod_{k=1}^n \frac{n^2 + k}{n^2 - k} = e.$$

(c) Rewrite the product in the equivalent form

$$\prod_{k=1}^n \frac{(n+k)a}{(n+k)a-1} = \prod_{k=1}^n \frac{1}{1 - \frac{1}{n} \cdot \frac{1}{a \left(1 + \frac{k}{n} \right)}}.$$

Let $f(x) = \frac{1}{1-x/a}$ and $g(x) = \frac{1}{1+x}$. Then

$$f'(0) = \frac{1}{a} \quad \text{and} \quad \int_0^1 g(x) dx = \ln 2.$$

So that by corollary 2,

$$\lim_{n \rightarrow \infty} \frac{(n+k)a}{(n+k)a-1} = e^{\frac{1}{a} \ln 2} = 2^{\frac{1}{a}}.$$

(d) Rewrite the expression in the equivalent form

$$\prod_{k=1}^n \cos \frac{ka}{n\sqrt{n}} = \prod_{k=1}^n \cos a \sqrt{\frac{1}{n} \left(\frac{k}{n} \right)^2}.$$

Let $f(x) = \cos a \sqrt{x}$ and $g(x) = x^2$. Then

$$f'(0) = -\frac{a^2}{2} \quad \text{and} \quad \int_0^1 g(x) dx = \frac{1}{3}.$$

So that by corollary 2

$$\lim_{n \rightarrow \infty} \prod_{k=1}^n \cos \frac{ka}{n\sqrt{n}} = e^{-\frac{a^2}{6}}.$$

If use the graduation agility for $[a, b]$, so it is also could use the theorem to solve many other types of limit cases.

Example 3 Evaluate of the following limit

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n^2}} \right).$$

Solution Let $f(x) = x$, $g(x) = \frac{1}{\sqrt{1+x^2}}$, make $[0, 1]$ into n^2 same parts and choose a point $\xi_k = \frac{k}{n^2}$

($k=1, 2, \cdots, n^2$) which in these parts, so

$$\sum_{k=1}^{n^2} f \left[\frac{1}{n^2} g \left(\frac{k}{n^2} \right) \right] = \sum_{k=1}^{n^2} \frac{1}{n^2} \frac{1}{\sqrt{1 + \frac{k}{n^2}}},$$

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{n^2} f\left[\frac{1}{n^2} g\left(\frac{k}{n^2}\right)\right] = \lim_{n \rightarrow \infty} \sum_{k=1}^{n^2} \frac{1}{n^2} \frac{1}{\sqrt{1 + \frac{k}{n^2}}} = f'(0) \int_0^1 g(x) dx = 2(\sqrt{2} - 1).$$

So

$$\sum_{k=1}^{n^2} \frac{1}{\sqrt{n^2 + k}} = n \sum_{k=1}^{n^2} \frac{1}{n^2} \frac{1}{\sqrt{1 + \frac{k}{n^2}}} \rightarrow \infty \quad (n \rightarrow \infty).$$

[References]

- [1] 王寿生等. 微积分解题方法与技巧[M]. 西安:西北工业大学出版社,1990.
- [2] 林源渠等. 数学分析习题集[M]. 北京:北京大学出版社,1993.
- [3] [美]波利亚等. 数学分析中的问题与定理[M]. 张奠宙等译. 上海:上海科技出版社,1985.
- [4] Loren C Larson. Problem-Solving Through Problems[M]. Printed and bound by R. R. Donnelley & Sons, Harrisonburg, Virginia. 175 Fifth Avenue, New York, New York 10010, U. S. A. Springer Verlag New York Inc., 1983.

极限的一个定理及其应用

袁南桥

(达县师范高等专科学校,四川 达州市 635000)

[摘 要] 给出了一个极限定理,能较好地解决一类特殊“和式”的极限问题.同时,利用对数函数的性质,又能够用来解决一些“积式”的极限.

[关键词] 极限定理;和式;积式